A Space-angle Discontinuous Galerkin Method for Two-Dimensional Radiative Transfer Equation with Reflective Boundary Conditions

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***Abstract*—The radiative transfer equation (RTE) for two-dimensional problems involving scattering, absorption and radiation with diffusely and specularly reflective boundaries is solved using the discontinuous Galerkin (DG) finite element method (FEM). Both space and angle directions are discretized by the DG method. A parallel procedure is used that solves a succession of sub-domains created by domain decomposition (DD) and angular decomposition (AD). Two problems are conducted with the reflection boundary conditions to study the performance of the method.**

I. Introduction

The propagation of radiation in the form of electromagnetic waves through a medium is affected by absorption, emission, and scattering processes. The radiative transfer equation (RTE) mathematically describes this interaction, which has a wide range of applications in such areas as heat transfer, neutron transport, atmospheric science, optical molecular imaging and some other applications. The discontinuous Galerkin (DG) finite element method (FEM) is one of the most popular grid-based numerical methods for solving the RTE due to its high order accuracy and flexibility in mesh grids. The basis functions used in the DG method are discontinuous across element interfaces; accordingly, the jump condition between interior traces of solution and the so-called numerical flux is weakly enforced on the interface boundaries. The space-angle DG methods that fully discretize the spatial and angular domain are specially suitable for the RTE, since the evolution of solution along characteristics can be strongly discontinuous when there are local radiation sources or radiation incidence coming from the boundary surface, especially the reflective ones. Previous work has proved that the space-angle DG method can be applied to solving the RTEs with high order precision in parallel with the domain decomposition (DD) and angular decomposition (AD) schemes [1], [2], [3], [4] . In this paper, a parallel space-angle DG method is used to solve the steady state radiative transfer problems with the diffuse and specular reflection boundary conditions in the 2D complex geometries.

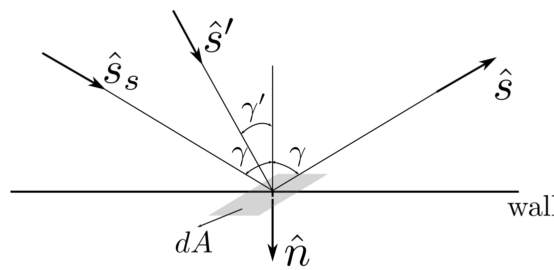

Fig. 1. A schematic of the radiative intensity reflected from a surface.

II. Formulation & Boundary Conditions

The steady-state radiative transfer equation for an emitting, absorbing, and anisotropically scattering medium in the two-dimensional Cartesian coordinate is written as,

$$\begin{aligned}&\sqrt{1-\mu^2}\cos\varphi\frac{\partial I}{\partial x}+\sqrt{1-\mu^2}\sin\varphi\frac{\partial I}{\partial y}+\beta I=\\&\kappa I_B(x,y)+\frac{\sigma_s}{4\pi}\int_{-\pi}^{\pi}\int_{-1}^{1}I\Phi(\mu,\varphi,\mu',\varphi')\mathrm{d}\mu'\mathrm{d}\varphi',\end{aligned} \tag{1}$$

where the unknown field I is the radiative intensity of four variables $I = I(x, y, \mu, \varphi)$, β is the extinction coefficient, κ is the absorption coefficient, σ_s is the scattering coefficient, $\Phi(\mu, \varphi, \mu', \varphi')$ is known as the phase function. Its boundary condition with diffuse and specular reflection on an opaque wall is,

$$I_w(\hat{\boldsymbol{s}})=\frac{\rho^d}{\pi}\int_{\hat{\boldsymbol{s}}'\cdot\hat{\boldsymbol{n}}>0}I_w(\hat{\boldsymbol{s}}')|\hat{\boldsymbol{s}}'\cdot\hat{\boldsymbol{n}}|\mathrm{d}\mu'\mathrm{d}\varphi'+\rho^s I_w(\hat{\boldsymbol{s}}_s), \tag{2}$$

where $I_w(\hat{\boldsymbol{s}})$ is the radiative intensity at the boundary in the solid angle direction $\hat{\boldsymbol{s}} = (\sqrt{1-\mu^2}\cos\varphi, \sqrt{1-\mu^2}\sin\varphi, \mu)$, $\hat{\boldsymbol{s}}'$ and $\hat{\boldsymbol{s}}_s$ are the diffuse and specular directions, respectively, $\hat{\boldsymbol{n}}$ is the outward surface normal, as depicted in Fig. 1, ρ^d and ρ^s are the diffuse and specular components of the reflectance, respectively.

In a DG formulation, residuals (errors) must be specified both in the interior and on the boundary of elements. The weighted residual (WR) of the finite element formulations

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formed by multiplying the RTE (Eqn. 1) by the weight function $\hat{H}$

$$\int_{Q} \hat{H}\left[\sqrt{1-\mu^2}\cos\varphi\frac{\partial I}{\partial x}+\sqrt{1-\mu^2}\sin\varphi\frac{\partial I}{\partial y}\right]\mathrm{d}V + \int_{Q}\hat{H}\left[\beta I-\kappa I_B(x,y)\right]\mathrm{d}V - \int_{Q}\hat{H}\left[\frac{\sigma_s}{4\pi}\oint_{4\pi} I\Phi(\mu,\varphi,\mu',\varphi')\mathrm{d}\mu'\mathrm{d}\varphi'd\Omega'\right]\mathrm{d}V + \int_{\partial Q}\hat{H}\left(I^*-I\right)\hat{\boldsymbol{s}}\cdot\hat{\boldsymbol{n}}\mathrm{d}A = 0, \quad (3)$$

where I^* is the target value in the DG formulation, Q is the element, ∂Q is the element boundary, The weight function $\hat{H}$ and trial solution I are polynomials of order p in both space and angle, interpolated with respect to a local coordinate system. The target value I^* corresponds to the upstream value along the direction of wave propagation where $\hat{\boldsymbol{s}}\cdot n<0$ is the inflow direction and $\hat{\boldsymbol{s}}\cdot n>0$ is the outflow direction. The diffuse and specular reflection boundary conditions are implemented differently. For the diffuse reflection boundary condition, the integral term in Eqn. (2) is calculated using Gauss quadrature. For the specular reflection boundary condition, the first step is to find the corresponding element of the specular direction. By interpolating the solution within the element at the specular direction, the reflective intensity is then obtained.

The computations are executed parallelly by using the domain and angular decomposition.

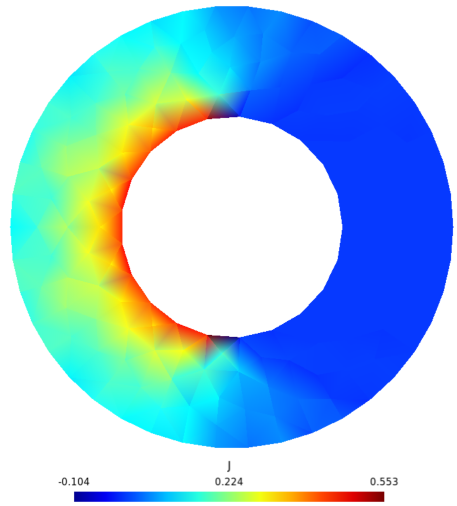
III. Numerical Examples

The diffuse reflection boundary condition is applied in the first example. A problem of radiation in an annular furnace with inner and outer radii of $R_1=1$ and $R_2=2$ is considered. The right part of the inner wall is cold ($I_w|_{x\geq 0}=0$) while the left part is hot ($I_w|_{x<0}=1$). The diffuse reflection boundary is set on the the outer wall with the diffuse reflectance $\rho^d=1.0$. The extinction, absorption and scattering coefficients are $\beta=0.05$, $\kappa=0$ and $\sigma_s=0$, respectively. The result of radiation density J is shown in Fig. 2(a).

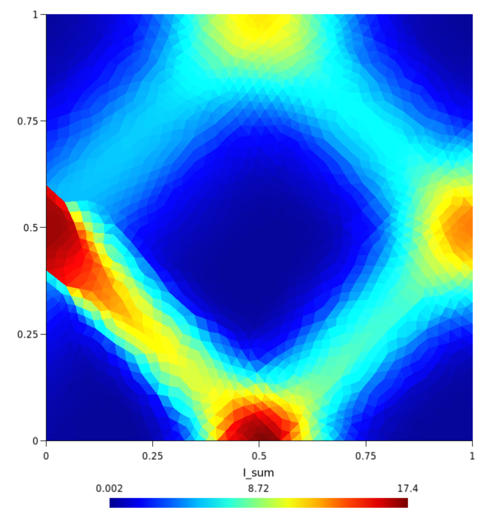
The second problem is an example of specular reflection boundary condition. In the 1 square geometry, an anisotropic prescribed incidence is set on the left wall. The rest walls are purely specularly reflective. The extinction, absorption and scattering coefficients are $\beta=0.01$, $\kappa=0$ and $\sigma_s=0$, respectively. The results of the summation of the radiative intensity over the quadrature points are shown in Fig. 2(b).

IV. Conclusions

We presented a space-angle DG method for the numerical solution of 2D radiative transfer problems in participating media. Both the diffusely and specularly reflective boundary conditions are considered. This application shows that the scheme can be used for the solution of complex radiative transfer problems.



(a) Example 1.



(b) Example 2.

Fig. 2. Contour plot of (a) the radiation density; (b) the summation of the radiative intensity over the solid angle.

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