A Space-angle Discontinuous Galerkin Method for One-Dimensional Cylindrical Radiative Transfer Equation with Angular Decomposition

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***Abstract*—The radiative transfer equation (RTE) for one-dimensional cylindrical problem involving scattering, absorption and radiation is solved using the discontinuous Galerkin (DG) finite element method (FEM). Both space and angle directions are discretized by the DG method. A special iterative procedure is used that solves a succession of sub-domains created by angular decomposition (AD). The problem is formulated for nonzero phase functions. The angular interaction terms are not explicitly added to the system solution matrix. A few benchmark problems are conducted to study the performance of the method.**

I. Introduction

The radiative transfer equation (RTE) describes the radiation intensity while propagating in an absorbing, scattering and emitting medium. The RTE plays an important role in radiation transfer in atmosphere, semitransparent liquid and solids, porous materials, and many other participating media. In order to acquire the radiative intensity field for a problem, different numerical methods have been studied for solving the RTE. One of the methods to numerically solve the RTE in both spatial and angular domains is the discontinuous Galerkin (DG) method. For a DG method the basis functions are discontinuous across element interfaces; accordingly the jump between interior traces of solution and the so-called numerical flux is weakly enforced on inter-element boundaries. The DG method is specifically suitable for the RTE, since the evolution of solution along characteristics can result in (strong) discontinuities in both space and angle. In our previous work, the RTE for a plane-parallel and two-dimensional axisymmetric problems are directly solved by the DG method [1], [2], [3], [4]. However, when dealing with a higher dimensional problem, a direct solver ends up with much higher memory usage. To overcome this problem, an iterative DG solver with the angular decomposition scheme is introduced in this paper to solve a one-dimensional cylindrical RTE problem.

II. Formulation & Iterative scheme

The general equation of radiative transfer for an emitting, absorbing, and anisotropically scattering medium is written as,

$$\hat{\boldsymbol{s}} \cdot \nabla I + \beta I = \frac{\sigma_s}{4\pi} \oint_{4\pi} I\left(\boldsymbol{r}, \hat{\boldsymbol{s}}'\right) \Phi\left(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}'\right) d\Omega' + q\left(\boldsymbol{r}, \hat{\boldsymbol{s}}\right) \quad (1)$$

where I is the spectral radiative intensity in the direction $\hat{\boldsymbol{s}}$ and the spatial position $\boldsymbol{r}$, β is the extinction coefficient, σ_s is the scattering coefficient, $\Phi\left(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}'\right)$ is known as the phase function, and q is the source term.

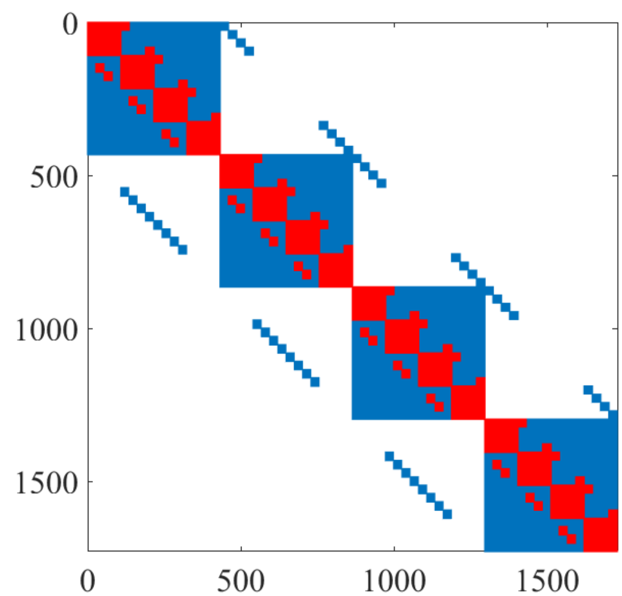

Fig. 1. Sparsity pattern of the stiffness matrices in a $4 \times 4 \times 4$ domain. The global stiffness is in blue; the sub-domain matrices are in red.

In a DG formulation, residuals (errors) must be specified both in the interior and on the boundary of elements. The weighted residual statement (WRS) of the finite element formulations formed by multiplying the RTE (Eqn. 1) by the weight function $\hat{H}$ is,

$$\begin{aligned} &\int_Q \hat{H}\left[\hat{\boldsymbol{s}} \cdot \nabla I + \beta I - q\right] \mathrm{d}V \\ &- \int_Q \hat{H}\left[\frac{\sigma_s}{4\pi} \oint_{4\pi} I\left(\boldsymbol{r}, \hat{\boldsymbol{s}}'\right) \Phi\left(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}'\right) d\Omega'\right] \mathrm{d}V, \\ &+ \int_{\partial Q} \hat{H}\left(I^* - I\right) \hat{\boldsymbol{s}} \cdot n \mathrm{d}A = 0 \end{aligned} \quad (2)$$

where I^* is the target value in the DG formulation, $\mathcal{Q}$ is the element, $\partial\mathcal{Q}$ is the element boundary, n is the normal vector of the element facet. The weight function $\hat{H}$ and trial solution I are polynomials of order p in both space and angle, interpolated with respect to a local coordinate system. The target value I^* corresponds to the upstream value along the direction of wave propagation where $\hat{\boldsymbol{s}} \cdot n < 0$ is the inflow direction and $\hat{\boldsymbol{s}} \cdot n > 0$ is the outflow direction. Each of the above terms are placed into a local stiffness and force tensor which is then transferred to a global stiffness $\underset{\sim}{K}$ matrix and force $\boldsymbol{F}$ vector in the equation: $\underset{\sim}{K}\boldsymbol{a} = \boldsymbol{F}$. The unknown vector $\boldsymbol{a}$ can be obtained by the direct solution of this linear system.

If the angular decomposition (AD) is applied, meaning the space-angle domain is sliced along the angle direction and divided into several sub-domains, each sub-domain can be solved separately. This significantly reduces the degrees of freedom within a slab and makes the stiffness sparser. For example, Fig. 1 shows the sparsity patterns of a global stiffness and sub-domain stiffness matrices. Subsequently, the domain is solved iteratively by updating the solution of each sub-domain until the residuals of the solution in all slabs converge to zero. Conceptually, the solution is expressed as a series,

$$\boldsymbol{a}_n = A^{-1}\left[\sum_{i=0}^{n}\left(-BA^{-1}\right)^i\right]\boldsymbol{F}, \tag{3}$$

where A is the global matrix from sub-domain stiffness matrices and B is the stiffness contributions by the angular integration terms in Eqn. 2. However, the solution does not converge if the spectral radius of BA^{-1} is greater than 1. Inspiring by the Newton-Raphson method, a relaxation factor α can be provided to help establish convergence for possibly divergent Eqn. (3). The solution of the modified iterative scheme is expressed as,

$$\boldsymbol{a}_n = A^{-1}\sum_{i=0}^{n}(-1)^i\left[(\alpha-1)\boldsymbol{I} + \alpha BA^{-1}\right]^i\boldsymbol{F}. \tag{4}$$

α is chosen small enough to ensure the spectral radius of $(\alpha-1)\boldsymbol{I} + \alpha BA^{-1}$ be less than 1. According to Eqn. 2, the WRS of 1D cylindrical RTE, depending on r in space and the cosine of the polar angle, $\mu = \cos\theta_s$ and $\tilde{\varphi}$ in angle, is derived easily by providing the formulation of $\hat{\boldsymbol{s}} \cdot \nabla I = \sin\theta\cos\tilde{\varphi}\frac{\partial I(r,\mu,\tilde{\varphi})}{\partial r} - \frac{\sin\theta\sin\tilde{\varphi}}{r}\frac{\partial I(r,\mu,\tilde{\varphi})}{\partial\tilde{\varphi}}$. The AD scheme is applied in the μ direction.

III. Numerical Examples

To validate the DG RTE solver, the Method of Manufactured Solution (MMS) is used. In the MMS, the source term in Eqn. (1) is set such that a chosen exact solution I^e satisfies it. If the manufactured solution space belongs to the space of finite element solution, i.e., when it is a polynomial of order equal or less than that used to interpolate the trial solution, the exact solution is recovered. For a simple example, I^e is given by, $I^e(r,\mu,\tilde{\varphi}) = -r\mu^2 + r^2\mu^2 + \frac{1}{2}\mu^2 - r\tilde{\varphi}^2 + r^2\tilde{\varphi}^2 + \frac{1}{2}\tilde{\varphi}^2 + \mu\tilde{\varphi} + \mu^2\tilde{\varphi}^2$. The phase function is simply set as, $\Phi = 1$. To ensure that the FEM elements capture the imposed solution above, the polynomial order of the basis function is set to $p_r = p_\mu = p_{\tilde{\varphi}} = 2$. Both the direct solver and the iterative solver capture the exact solutions.

To show the efficiency of the iterative solver, another example is investigated. In this example, inner and outer radii are $R_1 = 1$, $R_2 = 2$, $\beta = 1$, and $\sigma_s = 0.1$. $\Phi = 1$ corresponds to isotropic scattering. The inner surface is hot ($T_1 = 2000K$) and highly reflective ($\epsilon_1 = 0.1$); the outer surface is relatively cool ($T_2 = 400K$) and is a strong absorber ($\epsilon_2 = 0.9$). A $16\times16\times16$ grid with polynomial order $p = 1$ is used for the DG solution. For the iterative solver, the AD scheme is used to slice the domain in to 16 sub-domains.

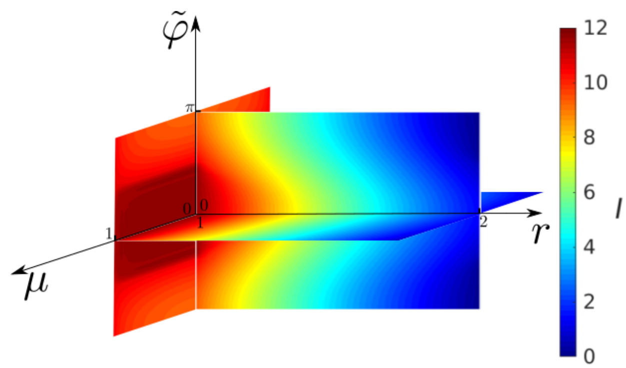

Fig. 2. Contour plot of radiation intensity of the benchmark problem for $R_1/R_2 = 0.5$ and $\tau_2 - \tau_1 = 10$.

The result is shown in Fig. 2. The direct solver takes the memory usage of about 16GB, whereas the iterative solver only uses about 2GB of memory.

IV. Conclusions

We presented an iterative DG method for the numerical solution of 1D cylindrical radiative transfer problem in participating media. Comparison has been made between the direct solver and the iterative solver. The iterative solver significantly makes the stiffness matrix sparser, thus, reduces the usage of memory. Moreover, it significantly reduces computational costs. We plan to employ the iterative scheme for high dimensional RTEs in strongly scattering media.

References

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