# A Parallel Space-angle Discontinuous Galerkin Method for Radiative Transfer in Two-dimensional Rectangular Enclosures

H. Wang, R. Abedi,
Department of Mechanical Aerospace and Biomedical Engineering
University of Tennessee Space Institute
Manchester, TN 37388-9700, USA
hwang@utsi.edu,rabedi@utsi.edu

S. Mudaliar
Sensors Directorate
Air Force Research Laboratory, Wright-Patterson AFB
Dayton, OH 45433-7318, USA
saba.mudaliar@us.af.mil

***Abstract*—The radiative transfer equation (RTE) for two-dimensional problem is solved using the parallel space-angle discontinuous Galerkin (DG) finite element method (FEM). To parallelize the solving process, both domain decomposition (DD) and angular decomposition (AD) methods are applied. These methods are validated against exact solutions. A few benchmark problems are investigated to study the performance of the parallel DG solver.**

## I. Introduction

The radiative transfer equation (RTE) describes radiative intensity in a medium affected by absorption, emission, and scattering processes. The difficulty of solving RTE either analytically or numerically is that it is an integro-differential partial different equation with multi-dimensions. Over the past few decades, several numerical techniques for solving the RTE have been introduced. These include, but are certainly not limited to, Monte Carlo methods, discrete-ordinate methods, spherical harmonics methods, spectral methods, finite difference methods, and finite element methods. Methods involving discrete ordinates have received particular attention in the literature, for their easy implementation and parallelization for large scale computation. In spite of their popularity, the discrete ordinate methods are not the only methods used to solve the RTE. Discontinuous Galerkin (DG) methods relax the continuity constraint of continuous finite element methods (FEM), where jump solution between elements not only is enforced weakly but also is based on wave propagation direction. In addition, the DG method has its natural advantage in parallel computing compared to the continuous FEMs for its weaker coupling of elements. Previous work has proved that the space-angle DG method can be applied to solving the RTEs with high order precision [1], [2], [3], [4] and has the potential to do large scale computation in parallel. In this paper, a parallel space-angle DG method with angular decomposition and domain decomposition is introduced to solve the steady state two-dimensional radiative transfer problems.

## II. Formulation & Parallel Implementation

For steady state two-dimensional radiative transfer problems, the radiative intensity turns out to be a function of four variables, $I = I(x, y, \mu, \varphi)$. The 2D RTE for an emitting, absorbing, and anisotropically scattering medium is written as,

$$\sqrt{1-\mu^2}\cos\varphi\frac{\partial I}{\partial x} + \sqrt{1-\mu^2}\sin\varphi\frac{\partial I}{\partial y} = -\beta I + \kappa I_B(x,y) + \frac{\sigma_s}{4\pi}\int_{-\pi}^{\pi}\int_{-1}^{1} I\Phi(\mu,\varphi,\mu',\varphi')\mathrm{d}\mu'\mathrm{d}\varphi', \tag{1}$$

where $\beta$ is the extinction coefficient, $\kappa$ is the absorption coefficient, $\sigma_s$ is the scattering coefficient, $\Phi(\hat{s}, \hat{s}')$ is known as the phase function.

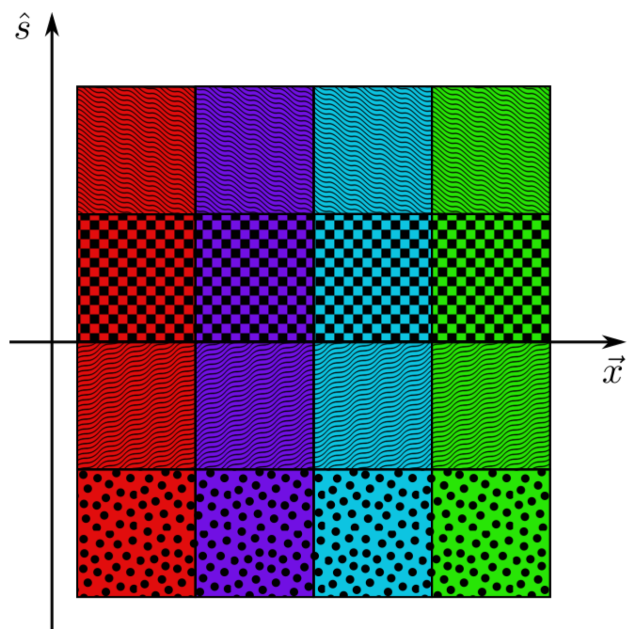


Fig. 1. A shcemetic of the DD method and AD method.

In a DG formulation, residuals (errors) must be specified both in the interior and on the boundary of elements. The weighted residual (WR) of the finite element formulations formed by multiplying the RTE (Eqn. 1) by the weight function $\hat{H}$

$$\begin{aligned}\int_{\mathcal{Q}} \hat{H}\left[\sqrt{1-\mu^2}\cos\varphi\frac{\partial I}{\partial x}+\sqrt{1-\mu^2}\sin\varphi\frac{\partial I}{\partial y}\right]\mathrm{d}V \\ +\int_{\mathcal{Q}}\hat{H}\left[\beta I-\kappa I_B(x,y)\right]\mathrm{d}V \\ -\int_{\mathcal{Q}}\hat{H}\left[\frac{\sigma_s}{4\pi}\oint_{4\pi} I\Phi(\mu,\varphi,\mu',\varphi')\mathrm{d}\mu'\mathrm{d}\varphi' d\Omega'\right]\mathrm{d}V \\ +\int_{\partial\mathcal{Q}}\hat{H}\left(I^*-I\right)\hat{\boldsymbol{s}}\cdot n\mathrm{d}A=0,\end{aligned} \tag{2}$$

where $I^*$ is the target value in the DG formulation, $\mathcal{Q}$ is the element, $\partial\mathcal{Q}$ is the element boundary, $n$ is the normal vector of the element facet. The weight function $\hat{H}$ and trial solution $I$ are polynomials of order $p$ in both space and angle, interpolated with respect to a local coordinate system. The target value $I^*$ corresponds to the upstream value along the direction of wave propagation where $\hat{\boldsymbol{s}}\cdot n<0$ is the inflow direction and $\hat{\boldsymbol{s}}\cdot n>0$ is the outflow direction.

To parallelize the 2D RTE solving process, either the spatial mesh or the extruded angular mesh is split into $n$ sub-domains. The former method is called domain decomposition (DD) and the latter one is called angular decomposition (AD) [5]. Fig. 1 illustrates the space and angle partitioning. For the DD method, the spatial mesh is divided into 4 sub-domains in different colors. The angular mesh in each sub-domain has to be same in order to communicate through processors. All the 4 sub-domains are assigned to a 4 MPI processes to solve the RTE in the sub-domains simultaneously. For each sub-domain interface, target values are the upstream values depending on the previous iteration from its neighbor sub-domain. In practice, after solving the RTE in the sub-domains, the solutions on the sub-domain interface are simply swapped. The iteration steps depend on the number of MPI processes. If 4 MPI processes are used, 3 iteration steps are required.

For the AD method, as shown in Fig. 1, the angular mesh is divided into 4 sub-domains in 4 different patterns. Similarly, all the 4 sub-domains are assigned to a 4 MPI processes to solve the RTE in the sub-domains simultaneously. However, if the angular integration involved, instead of swapping the solution on interfaces, a shared memory of accessing the information at each quadrature point is required and the RTE is solved iteratively.

## III. Numerical Examples

The Method of Manufactured Solution (MMS) is used to validate the parallel 2D RTE DG solver. In the MMS, an exact solution is given as an extra source term in the DG formulation. If the manufactured solution space belongs to the space of finite element solution, i.e., when it is a polynomial of order equal or less than that used to interpolate the trial solution, $I$, the exact solution is recovered. Both the DD method and the AD method capture the exact solutions.

A benchmark problem of an anisotropically scattering medium in a rectangular enclosure is investigated. The size of the rectangle is $1\times 1$. The incident radiation on the left boundary is $\bar{I}=1$. On the rest boundaries, the intensity remains 0. The Rayleigh phase function is employed in this problem. The extinction coefficient is $\beta=1$. The scattering coefficient is $\sigma_s=0.5$. The result is shown in Fig. 2 in different directions.

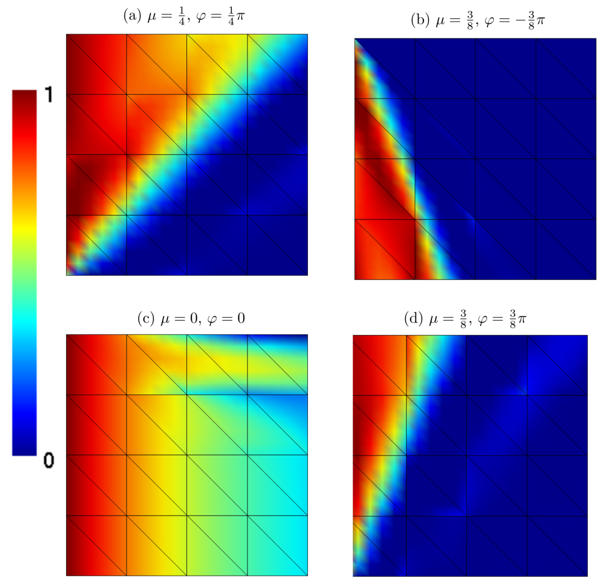


Fig. 2. Contour plot of radiation intensity of the benchmark problem at four different directions.

## IV. Conclusions

This paper has presented a parallel DG method for the numerical solution of a 2D radiative transfer problem with both DD method and AD method. The parallel DG method is essential and promising when solving a large system. To combine the DD method and AD method is our future work in order to make the space-angle DG method fully parallel.

## References

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