The Minimum Residual Interpolation Method applied to Multiple Scattering in MM-PO

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1 Introduction

In several applications it is important to solve a linear system of equations for multiple right hand sides. Assume a dense linear system with N unknowns and M right hand sides. With Gaussian elimination the solution time is $\mathcal{O}(N^3)$ for the factorization and $\mathcal{O}(MN^2)$ for the substitution part. For integral equations we can use fast matrix vector algorithms like the multilevel Fast Multipole Method (MLFMM) [1] together with iterative methods to reduce the solution time to $\mathcal{O}(KMN \log N)$. Here, K is the average number of iterations for one right hand side. When K and M are large the advantage of the fast method is not so clear because of the large constant in front of the scaling. A direct solver could be preferred in this case if the matrices fit into memory. This is the reason we developed the Minimum Residual Interpolation method (MRI) in [2]. MRI is efficient when the right hand sides depend smoothly on a parameter. The advantage of MRI is the independence of the underlying solution method and the ability to predict the residual without computing a matrix vector product. Once enough right hand sides are solved, MRI can predict the solutions to the remaining right hand sides.

In this paper the versatility of MRI is demonstrated. MRI is applied to the solution of the equations in the Method of Moment (MM), the Physical Optics method (PO) and the iterative Method of Moment - Physical Optics hybrid (MM-PO) described in [3]. The PO part is formulated as a Galerkin problem as in [3]. The MM part is handled with an iterative method that uses MLFMM matrix vector multiplication [4]. The coupling between the two regions is also handled with MLFMM. The iteration between MM and PO can be viewed as an iterative block Gauss-Seidel method.

2 Minimum Residual Interpolation method

The systems of linear equations that should be solved are

$$\mathbf{A}\mathbf{x}_i = \mathbf{b}_i,\ i = 1 \ldots M,\ \mathbf{A} \in \mathbb{C}^{N\times N},\ \mathbf{x}_i, \mathbf{b}_i \in \mathbb{C}^N. \tag{1}$$

The residual is defined as $\mathbf{r}_i = \mathbf{b}_i - \mathbf{A}\mathbf{x}_i$. An iterative method can solve the equations such that $\|\mathbf{r}_i\| \leq \varepsilon$ for some ε.

Assume that the solutions to $m < M$ right hand sides are known and are linearly independent. Let $\mathbf{s}_i$, $\mathbf{S}_m$, and $\mathbf{X}_m$ be defined by

$$\mathbf{A}\mathbf{x}_i = \mathbf{b}_i - \mathbf{r}_i \equiv \mathbf{s}_i,\ i = 1 \ldots m,\ \mathbf{X}_m = [\mathbf{x}_1\,\mathbf{x}_2\ \ldots\ \mathbf{x}_m]\,,\ \mathbf{S}_m = [\mathbf{s}_1\,\mathbf{s}_2\ \ldots\ \mathbf{s}_m]\,. \tag{2}$$

Let the $\mathbf{QR}$-decomposition of $\mathbf{S}_m$ be given by $\mathbf{A}\mathbf{X}_m = \mathbf{S}_m = \mathbf{Q}_{\mathbf{S}_m}\mathbf{R}_{\mathbf{S}_m}$. Then, one can compute a minimum residual solution such that [2]

$$\mathbf{x}_{m+1}^{(0)} = \mathbf{X}_m \mathbf{R}_{\mathbf{S}_m}^{-1} \mathbf{Q}_{\mathbf{S}_m}^{H} \mathbf{b}_{m+1},\ \mathbf{r}_{m+1}^{(0)} = (\mathbf{I} - \mathbf{Q}_{\mathbf{S}_m}\mathbf{Q}_{\mathbf{S}_m}^{H})\mathbf{b}_{m+1}. \tag{3}$$

If $\|\mathbf{r}_{m+1}^{(0)}\| > \varepsilon$ the solution $\mathbf{x}_{m+1}^{(0)}$ can be used in an iterative method as initial guess.

In [2] we proved for the case of electromagnetic plane wave scattering that $\|\mathbf{r}_{m+1}^{(0)}\| \approx C\,(\kappa d \Delta\phi)^m$ for some constant C. Here, κ is the wavenumber, d is the size of the object and $\Delta\phi$ is the difference in the spherical angle between adjacent plane waves. In a plane, it means that when $m \approx \kappa d/2$ the residual of the initial guess for any remaining right hand side is sufficiently small and that no further iterations are needed in the iterative method. For a general $\mathbf{b}_i = \mathbf{b}\,(\phi_i)$ and $\phi_i = i\Delta\phi$ a similar result was obtained.

3 MM-PO hybrid

Consider the case of electromagnetic scattering from a perfect electric conductor (PEC). To find the equivalent currents on the scatterer we formulate CFIE [1]. In the iterative MM-PO hybrid the scatterer is divided into a MM region Γ_1 where CFIE is solved and a PO region Γ_2 where an approximation to MFIE is solved. The PO approximation is obtained by assuming that the double integral in MFIE is zero. The PO region is divided into two parts: the lit region Γ_2^{L} and the shadowed region Γ_2^{S} where the electric current is assumed to be zero. The RWG basis function for triangles is used to discretize the problem. This leads to the equation

$$\begin{pmatrix} \mathbf{Z}_{11} & \mathbf{Z}_{12} \\ \mathbf{Z}_{21} & \mathbf{Z}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{pmatrix}. \tag{4}$$

The matrices $\mathbf{Z}_{ii}$ correspond to self-interactions within the two regions while $\mathbf{Z}_{12}$ and $\mathbf{Z}_{21}$ correspond to the interactions between the regions.

The solution to equation (4) is obtained from an iterative block Gauss-Seidel method

$$\begin{cases} \mathbf{I}_2^{(k)} = \mathbf{Z}_{22}^{-1}\left(\mathbf{V}_2 - \mathbf{Z}_{21}\mathbf{I}_1^{(k-1)}\right) \\ \mathbf{I}_1^{(k)} = \mathbf{Z}_{11}^{-1}\left(\mathbf{V}_1 - \mathbf{Z}_{12}\mathbf{I}_2^{(k)}\right) \end{cases} \tag{5}$$

where $\mathbf{I}^{(0)}$ is given by MRI. Both equations are solved with iterative methods that use $\mathbf{I}_i^{(k-1)}$ as initial guess. Convergence is usually achieved in a few iterations. The matrix vector multiplications with $\mathbf{Z}_{11}$, $\mathbf{Z}_{12}$ and $\mathbf{Z}_{21}$ are computed

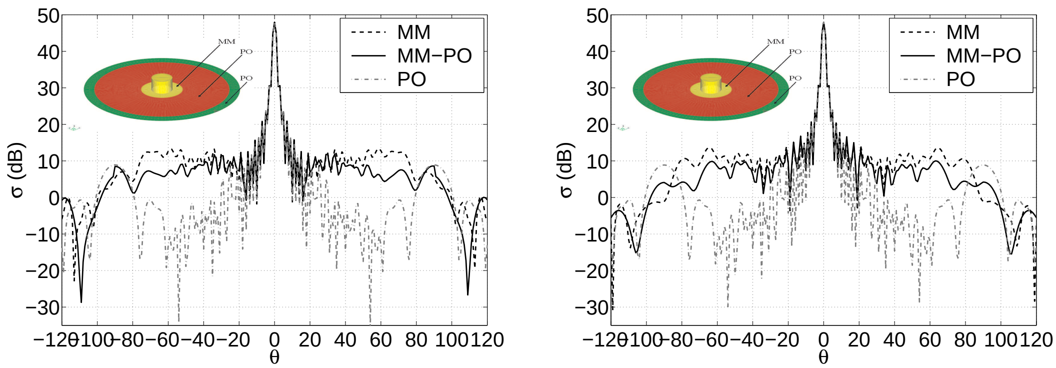

Figure 1: Solutions for the different methods. The x-z-polarized wave to the left and the y-polarized wave to the right.

with MLFMM, while $\mathbf{Z}_{22}$ is a sparse matrix with 5 entries on each row. Usually the number of unknowns in the PO region $N_{\mathrm{PO}} \gg N_{\mathrm{MM}}$ the number of unknowns in the MM region. The MM-part is preconditioned with SPAI [4]. The complexity of the method is $C_1 K_1 N_{\mathrm{MM}} \log N_{\mathrm{MM}} + C_1 K_2 N_{\mathrm{PO}} \log N_{\mathrm{PO}} + C_2 K_3 N_{\mathrm{PO}}$, where K_1 is the total number of iterations in Γ_{MM}, K_2 is the number of iterations in (5) and K_3 is the total number of iterations in Γ_{PO}. Usually K_2 is smaller than K_1 and K_3. The first or second part will dominate the total cost depending on the size of N_{MM}. The cost should be compared to $C_1 K N \log N$, the cost of using MM on the entire surface. Since one can expect that $K_1 < K_2 K$ the MM-PO hybrid is faster for cases when K_2 is $\mathcal{O}(1)$.

4 Results

The method is verified by computing the monostatic Radar Cross Section (RCS) from a cylinder mounted on a circular disc in the x-y-plane. At the frequency 750 MHz the cylinder is 2λ high and the radius is 1λ. The radius of the circular disc is 7.5λ. The discretization yielded about 37000 unknowns. The monostatic RCS is computed with MM, PO and the MM-PO hybrid. Since the object is open, EFIE is used in the MM-part. In the MM-PO hybrid the cylinder and a circular disc with radius two wavelengths is discretized with MM while the rest of the object is discretized with PO yielding $N_{MM} \approx N_{PO}/6$. MRI is used in all cases to decrease the solution time. The plane wave impinges in the x-z-plane. Both the x-z-polarized incident wave solution and the y-polarized incident wave solution were computed. A total of 482 solutions were computed. The iterations were stopped when $\|\mathbf{r}_i\| \leq 10^{-3} \|\mathbf{b}_i\|$.

The solutions obtained with the different methods are plotted in Figure 1. To compare the accuracy of the different methods the difference between the solution obtained with MM is compared to the other solutions in Figure 2. Clearly the MM-PO hybrid solution is more accurate than the pure PO solution. The PO method was 26 times faster than the MM-PO hybrid and 262 times faster than the MM method. MRI reduced the solution time compared

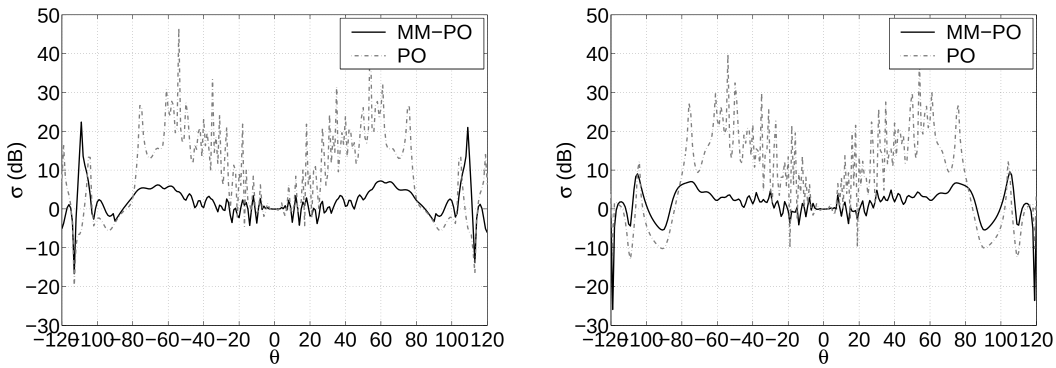

Figure 2: The difference between the MM solution and the other methods. The x-z-polarized wave to the left and the y-polarized wave to the right.

to solving one right hand side at a time by a factor 1.8 in the PO case, 8.7 in the MM-PO case and 6.8 in the MM case. After 119 right hand sides were solved by MM the solutions to the remaining right hand sides were obtained by just using MRI. Since both polarizations were solved we expected $m \approx \kappa d \approx 134$ from theory. MRI only used 0.2 % of the total solution time. The MM-PO hybrid needed 137 solutions and MRI used 2 % of the total time, while PO needed 149 solutions and MRI used 61 % of the solution time. That more solutions were needed in these cases is explained by the fact that we apply shadowing, which makes the right hand side less smooth.

In conclusion MRI is applied to electromagnetic scattering. An example is given where it is demonstrated that MRI reduces the solution time.

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